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HW 6

Solutions

① (a) Note that $\vec{v} = 2\vec{u}$

So, $1\cdot\vec{v} - 2\vec{u} = \vec{0}$

Thus, $\vec{u}, \vec{v}$ are lin. dep.

We have that $\vec{v} = 2\vec{u}$ is a lin. combination of $\vec{u}$.

① (b) We want to find the solutions to

$$c_1\vec{v} + c_2\vec{u} + c_3\vec{w} = \vec{0}$$

which becomes

$$c_1\langle 1,-1\rangle + c_2\langle 0,-3\rangle + c_3\langle 2,1\rangle = \langle 0,0\rangle$$

which gives

$$\langle c_1,-c_1\rangle + \langle 0,-3c_2\rangle + \langle 2c_3, c_3\rangle = \langle 0,0\rangle$$

which gives

$$\langle c_1 + 2c_3, -c_1 - 3c_2 + c_3\rangle = \langle 0, 0\rangle$$

Thus,

$$\boxed{\begin{aligned} c_1 \quad\quad\; + 2c_3 &= 0 \\ -c_1 - 3c_2 + c_3 &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & -3 & 1 & 0 \end{array}\right) \xrightarrow{R_1 + R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right)$$

So we get:

$$\boxed{\begin{aligned} c_1 \quad\quad + 2c_3 &= 0 \\ c_2 - c_3 &= 0 \end{aligned}}$$

leading: c_1, c_2
free: c_3

$$\longrightarrow \boxed{\begin{aligned} c_1 &= -2c_3 \\ c_2 &= c_3 \\ c_3 &= t \end{aligned}}$$

3 $c_3 = t$

2 $c_2 = c_3 = t$

1 $c_1 = -2c_3 = -2t$

Plugging this back into $c_1\vec{v} + c_2\vec{u} + c_3\vec{w} = \vec{0}$ gives

$$-2t\vec{v} + t\vec{u} + t\vec{w} = \vec{0}$$

for any t.

For example if $t = 1$ we get

$$-2\vec{v} + \vec{u} + \vec{w} = \vec{0}$$

The vectors are linearly dependent and for example we can write $\vec{u}$ as a linear combination of $\vec{v}$ and $\vec{w}$ as follows:

$$\vec{u} = 2\vec{v} - \vec{w}$$

$\boxed{①\,(c)}$

We want to find the solutions to

$$c_1\vec{v} + c_2\vec{u} + c_3\vec{w} = \vec{0}.$$

This becomes

$$c_1\langle 2,-1,3\rangle + c_2\langle 4,1,2\rangle + c_3\langle 8,-1,8\rangle = \langle 0,0,0\rangle$$

which gives

$$\langle 2c_1,-c_1,3c_1\rangle + \langle 4c_2,c_2,2c_2\rangle + \langle 8c_3,-c_3,8c_3\rangle = \langle 0,0,0\rangle$$

which gives

$$\langle \underbrace{2c_1+4c_2+8c_3}, \underbrace{-c_1+c_2-c_3}, \underbrace{3c_1+2c_2+8c_3}\rangle = \langle 0,0,0\rangle$$

This gives

$$\boxed{\begin{aligned} 2c_1 + 4c_2 + 8c_3 &= 0 \\ -c_1 + c_2 - c_3 &= 0 \\ 3c_1 + 2c + 8c_3 &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{ccc|c} 2 & 4 & 8 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & 2 & 8 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_1 \to R_1} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ -1 & 1 & -1 & 0 \\ 3 & 2 & 8 & 0 \end{array}\right) \xrightarrow[-3R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & -4 & -4 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{-1}{4}R_3 \to R_3]{\frac{1}{3}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \xrightarrow{-R_2+R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We get

$$\boxed{\begin{aligned} c_1 + 2c_2 + 4c_3 &= 0 \quad (1) \\ c_2 + c_3 &= 0 \quad (2) \end{aligned}}$$

leading: c_1, c_2
free: c_3

So,

$$\boxed{\begin{aligned} c_1 &= -2c_2 - 4c_3 \quad (1) \\ c_2 &= -c_3 \quad (2) \\ c_3 &= t \quad (3) \end{aligned}}$$

Thus,

(3) $c_3 = t$

(2) $c_2 = -c_3 = -t$

(1) $c_1 = -2c_2 - 4c_3 = -2(-t) - 4t = -2t$

Plugging this back into $c_1\vec{v} + c_2\vec{u} + c_3\vec{w} = \vec{0}$ gives

$$-2t\vec{v} - t\vec{u} + t\vec{w} = \vec{0}$$

for any t. If we plug in $t = 1$ we get

$$-2\vec{v} - \vec{u} + \vec{w} = \vec{0}$$

So, $\vec{v}, \vec{u}, \vec{w}$ are linearly dependent and for example we can express $\vec{w}$ as a linear combination of $\vec{v}$ and $\vec{u}$ as follows:

$$\vec{w} = 2\vec{v} + \vec{u}$$

$\boxed{2(a)}$

Solving $c_1\vec{a} + c_2\vec{b} = \vec{0}$ we get

$$c_1\langle 1,1\rangle + c_2\langle -1,1\rangle = \langle 0,0\rangle$$

which gives

$$\langle c_1 - c_2, c_1 + c_2\rangle = \langle 0, 0\rangle$$

which gives

$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \\ c_1 + c_2 &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{-R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

This gives:

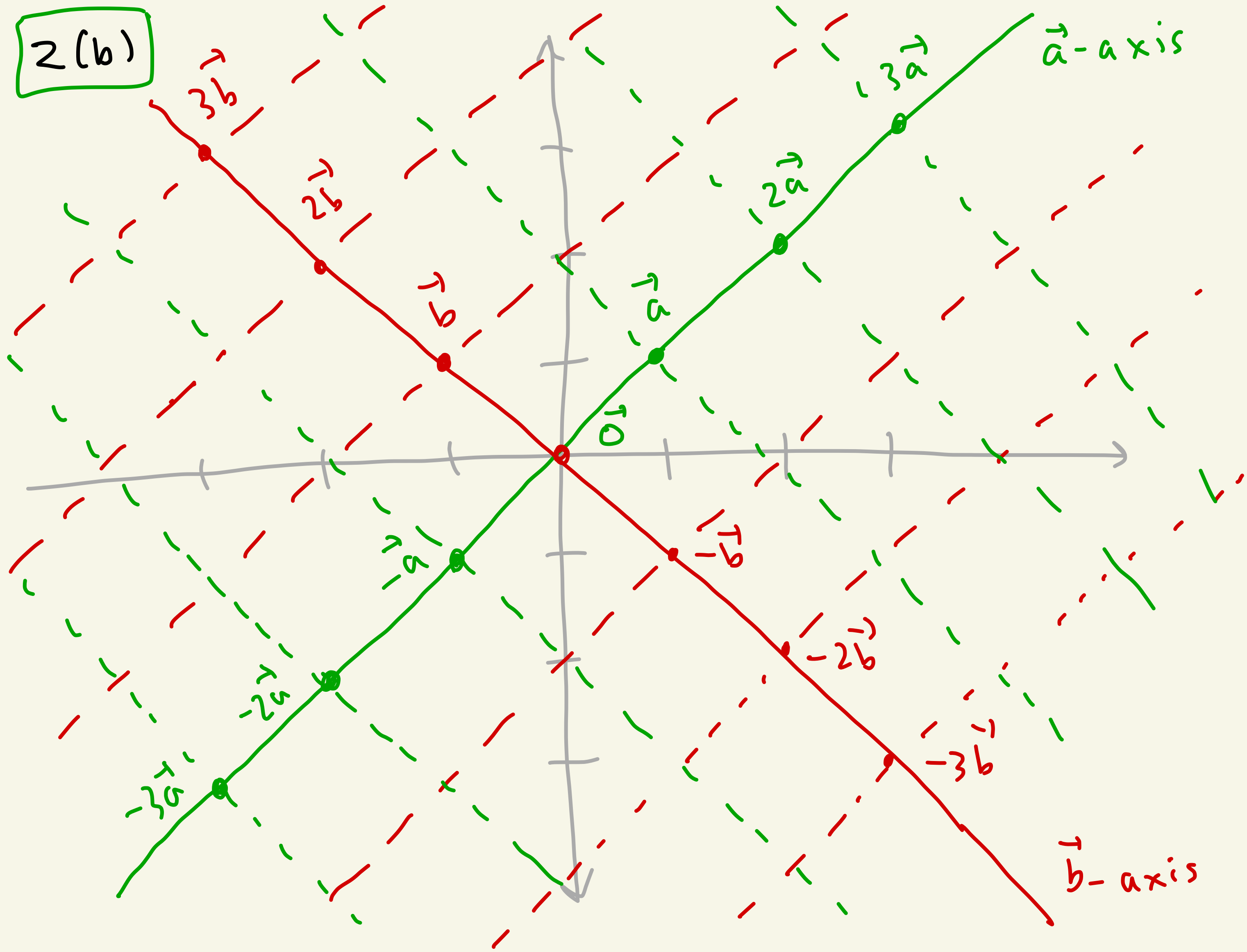
$$\boxed{\begin{aligned} c_1 - c_2 &= 0 \quad (1) \\ c_2 &= 0 \quad (2) \end{aligned}}$$

So, (2) $c_2 = 0$. And (1) $c_1 = c_2 = 0$.

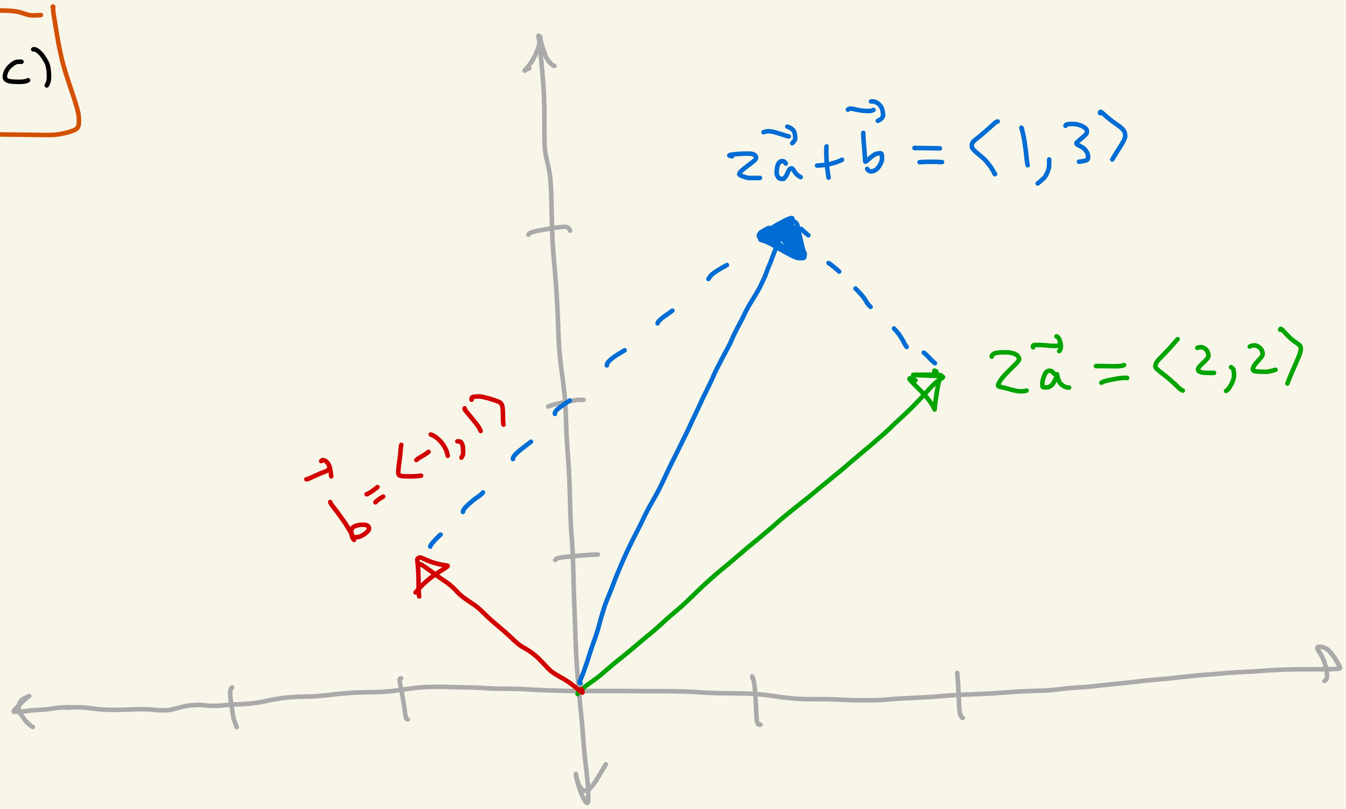
Thus, the only solution to $c_1\vec{a} + c_2\vec{b} = \vec{0}$ is $c_1 = 0, c_2 = 0$. So, $\vec{a}$ and $\vec{b}$ are linearly independent.

Since we have 2 linearly independent vectors in $\mathbb{R}^2$, $\beta = [\vec{a}, \vec{b}]$ is a basis for $\mathbb{R}^2$.

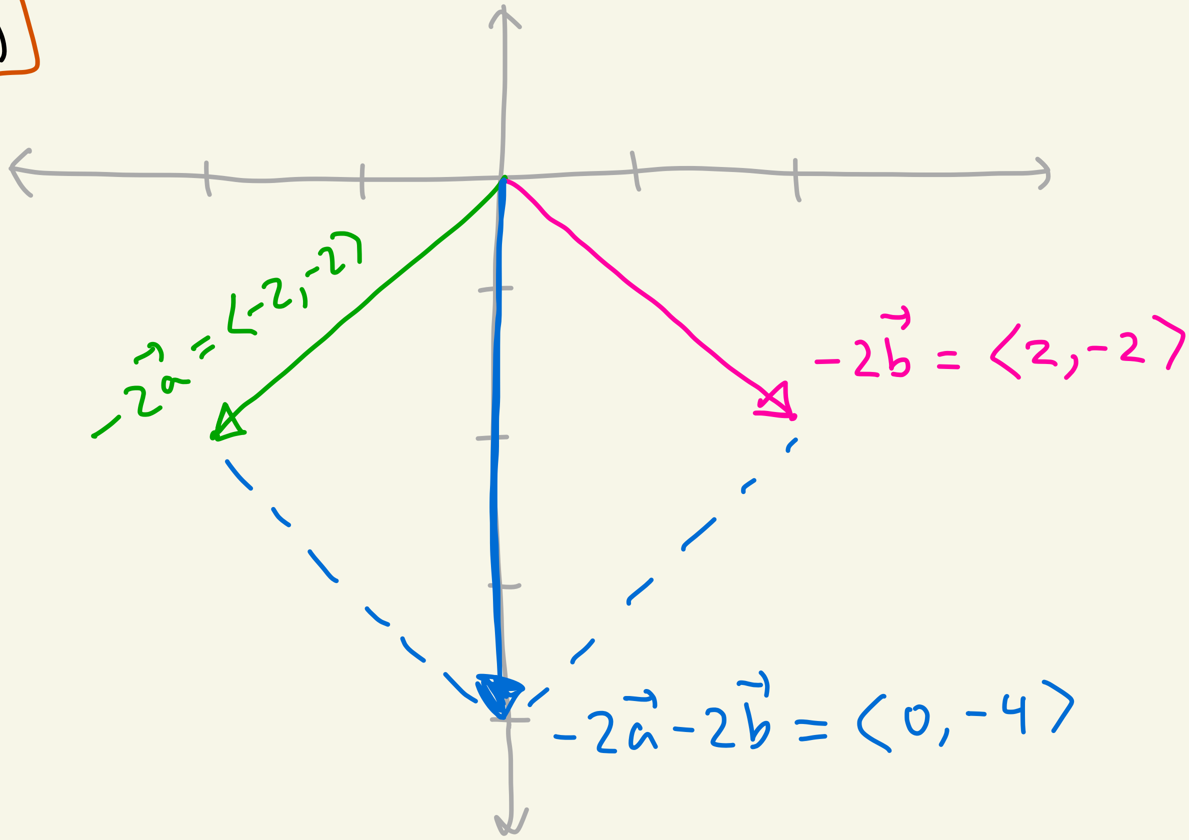
2(b)


$\vec{a}$-axis
$3\vec{a}$
$2\vec{a}$
$\vec{a}$
$\vec{0}$
$-\vec{a}$
$-2\vec{a}$
$-3\vec{a}$
$3\vec{b}$
$2\vec{b}$
$\vec{b}$
$-\vec{b}$
$-2\vec{b}$
$-3\vec{b}$
$\vec{b}$-axis

2(c)


$2\vec{a}+\vec{b} = \langle 1,3 \rangle$
$2\vec{a} = \langle 2,2 \rangle$
$\vec{b} = \langle -1,1 \rangle$

2 (d)



2 (e) We want to solve

$$\vec{v} = c_1 \vec{a} + c_2 \vec{b}$$

which gives

$$\langle -1,5 \rangle = c_1 \langle 1,1 \rangle + c_2 \langle -1,1 \rangle$$

which gives

$$\langle -1,5 \rangle = \langle c_1 - c_2, c_1 + c_2 \rangle$$

which gives

$$\boxed{\begin{aligned} c_1 - c_2 &= -1 \\ c_1 + c_2 &= 5 \end{aligned}}$$

Solving:

$$\left(\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & 5 \end{array}\right) \xrightarrow{-R_1+R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 2 & 6 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 1 & 3 \end{array}\right)$$

So,

$$\boxed{\begin{aligned} c_1 - c_2 &= -1 \\ c_2 &= 3 \end{aligned}} \longrightarrow \boxed{\begin{aligned} &c_2 = 3 \\ &c_1 = -1 + c_2 = -1 + 3 = 2 \end{aligned}}$$

Thus, plugging back into $\vec{v} = c_1\vec{a} + c_2\vec{b}$ gives

$\vec{v} = 2\vec{a} + 3\vec{b}$.

So,

$$\boxed{[v]_\beta = \langle 2, 3 \rangle}$$

$\boxed{\text{②(f)}}$ We want to solve $\vec{w} = c_1\vec{a} + c_2\vec{b}$.

This gives $\langle -3, -1 \rangle = c_1\langle 1, 1 \rangle + c_2\langle -1, 1 \rangle$.

So, $\langle -3, -1 \rangle = \langle c_1 - c_2, c_1 + c_2 \rangle$.

Thus, $\boxed{\begin{aligned} c_1 - c_2 &= -3 \\ c_1 + c_2 &= -1 \end{aligned}}$

Solving we get

$$\left(\begin{array}{cc|c} 1 & -1 & -3 \\ 1 & 1 & -1 \end{array}\right) \xrightarrow{-R_1 + R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 2 & 2 \end{array}\right) \xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{cc|c} 1 & -1 & -3 \\ 0 & 1 & 1 \end{array}\right)$$

So,

$$\boxed{\begin{aligned} c_1 - c_2 &= -3 \\ c_2 &= 1 \end{aligned}} \longrightarrow \boxed{\begin{aligned} &c_2 = 1 \\ &c_1 = -3 + c_2 = -3 + 1 = -2 \end{aligned}}$$

Thus, plugging back into $\vec{w} = c_1\vec{a} + c_2\vec{b}$ gives

$\vec{w} = -2\vec{a} + \vec{b}$. Thus, $\boxed{[\vec{w}]_\beta = \langle -2, 1 \rangle}$

2 (g)

$\vec{a}\cdot\vec{b} = \langle 1,1\rangle \cdot \langle -1,1\rangle = -1+1=0.$

So, β is orthogonal.

However,

$$\left.\begin{aligned} \|\vec{a}\| &= \sqrt{1^2+1^2} = \sqrt{2} \\ \|\vec{b}\| &= \sqrt{(-1)^2+1^2} = \sqrt{2} \end{aligned}\right\} \text{not length 1 vectors}$$

So, β is not orthonormal

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2 (h) Since β is an orthogonal basis we have

$$\vec{v} = \left(\frac{\vec{v}\cdot\vec{a}}{\|\vec{a}\|^2}\right)\vec{a} + \left(\frac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

we have:

$$\frac{\vec{v}\cdot\vec{a}}{\|\vec{a}\|^2} = \frac{\langle 10,\frac{1}{2}\rangle\cdot\langle 1,1\rangle}{\left(\sqrt{1^2+1^2}\right)^2} = \frac{10+\frac{1}{2}}{2} = \frac{21}{4}$$

and

$$\frac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|^2} = \frac{\langle 10,\frac{1}{2}\rangle\cdot\langle -1,1\rangle}{\left(\sqrt{(-1)^2+1^2}\right)^2} = \frac{-10+\frac{1}{2}}{2} = \frac{-19}{4}$$

Thus,

$$\underbrace{\langle 10,\tfrac{1}{2}\rangle}_{\vec{v}} = \underbrace{\frac{21}{4}\langle 1,1\rangle - \frac{19}{4}\langle -1,1\rangle}_{c_1\vec{a}+c_2\vec{b}}.$$

$\vec{v} = c_1\vec{a} + c_2\vec{b}$

So, $\boxed{[\vec{v}]_\beta = \langle \frac{21}{4}, \frac{-19}{4}\rangle}$

2 (i) Since β is an orthogonal basis we have

$$\vec{v} = \left(\frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2}\right)\vec{a} + \left(\frac{\vec{v} \cdot \vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$$

we have:

$$\frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2} = \frac{\langle 1,2\rangle \cdot \langle 1,1\rangle}{\left(\sqrt{1^2+1^2}\right)^2} = \frac{1+2}{2} = \frac{3}{2}$$

and

$$\frac{\vec{v} \cdot \vec{b}}{\|\vec{b}\|^2} = \frac{\langle 1,2\rangle \cdot \langle -1,1\rangle}{\left(\sqrt{(-1)^2+1^2}\right)^2} = \frac{-1+2}{2} = \frac{1}{2}$$

Thus,

$$\underbrace{\langle 1,2\rangle}_{\vec{v}} = \underbrace{\frac{3}{2}\langle 1,1\rangle + \frac{1}{2}\langle -1,1\rangle}_{c_1\vec{a} + c_2\vec{b}}.$$

So, $\boxed{[\vec{v}]_\beta = \langle \frac{3}{2}, \frac{1}{2}\rangle}$

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2 (j) Since $[\vec{v}]_\beta = \langle 5,-4\rangle$ we know that $\vec{v} = 5\vec{a} - 4\vec{b}$.

Thus, $\vec{v} = 5\langle 1,1\rangle - 4\langle -1,1\rangle = \langle 5+4, 5-4\rangle = \langle 9,1\rangle$

So, $\boxed{\vec{v} = \langle 9,1\rangle}$

3(a) We need to solve $c_1\vec{a}+c_2\vec{b}=\vec{0}$.

This is $c_1\langle 1,1\rangle + c_2\langle 1,0\rangle = \langle 0,0\rangle$.

This gives $\langle c_1+c_2, c_1\rangle = \langle 0,0\rangle$

So, $\boxed{\begin{aligned} c_1+c_2&=0 \\ c_1 \quad &=0 \end{aligned}}$ 1 2

This isn't reduced but its clear what the only sol. is.

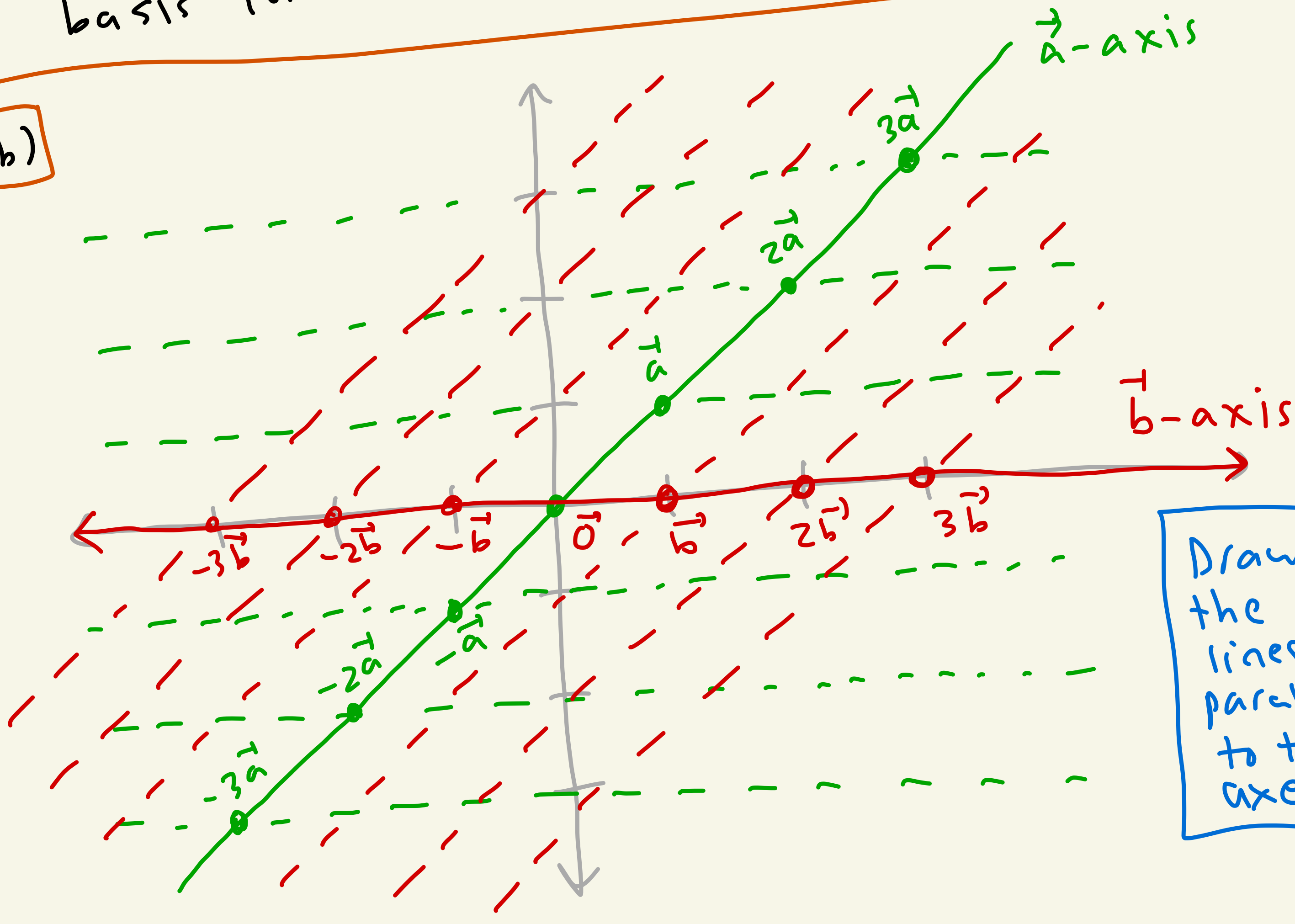
Thus 2 $c_1=0$. And $0+c_2=0 \rightarrow c_2=0$.

The only solution is $c_1=0, c_2=0$.

Thus, $\vec{a},\vec{b}$ are linearly independent.

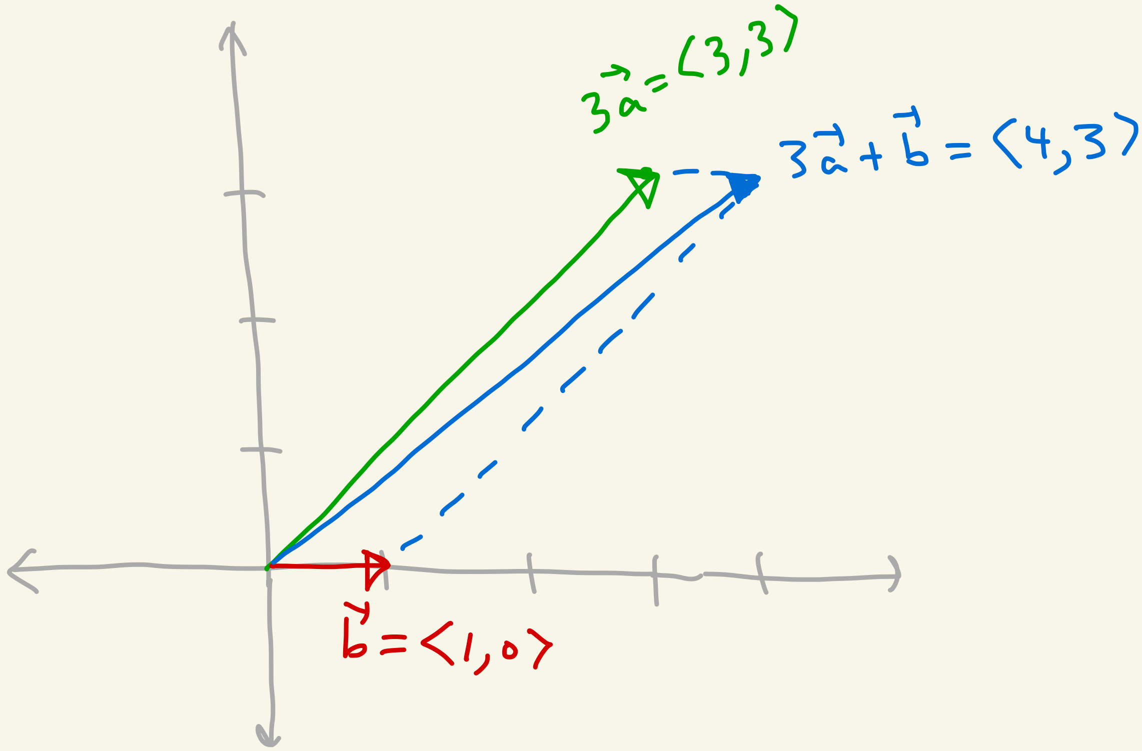
Since we have 2 linearly independent vectors in $\mathbb{R}^2$, we know that $\beta=[\vec{a},\vec{b}]$ is a basis for $\mathbb{R}^2$.

3(b)

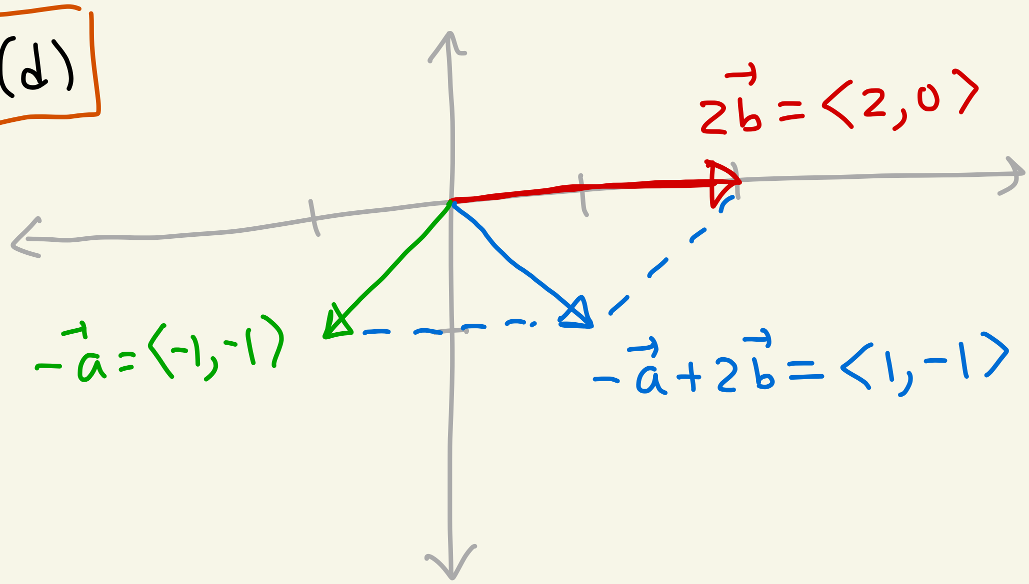


Draw the lines parallel to the axes

3 (c)



3 (d)



3 (e) Need to solve $\vec{v} = c_1\vec{a} + c_2\vec{b}$.

This is $\langle 1,2\rangle = c_1\langle 1,1\rangle + c_2\langle 1,0\rangle$.

This gives $\langle 1,2\rangle = \langle c_1+c_2, c_1\rangle$

So, $c_1+c_2 = 1$, $c_1 = 2$.

Thus, $c_1 = 2$, $c_2 = -1$.

So, $\vec{v} = 2\vec{a} - \vec{b}$

So, $\boxed{[\vec{v}]_\beta = \langle 2, -1 \rangle}$

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3 (f) Need to solve $\vec{w} = c_1\vec{a} + c_2\vec{b}$.

This is $\langle -1, 3 \rangle = c_1 \langle 1, 1 \rangle + c_2 \langle 1, 0 \rangle$

This gives $\langle -1, 3 \rangle = \langle c_1 + c_2, c_1 \rangle$.

Thus, $c_1 + c_2 = -1$ and $c_1 = 3$.

So, $c_1 = 3$, $c_2 = -4$.

Hence, $\vec{w} = 3\vec{a} - 4\vec{b}$.

So, $\boxed{[\vec{w}]_\beta = \langle 3, -4 \rangle}$

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3 (g)

$\vec{a} \cdot \vec{b} = \langle 1, 1 \rangle \cdot \langle 1, 0 \rangle = 1 + 0 = 1$

Since $\vec{a} \cdot \vec{b} \neq 0$, β is not orthogonal.

Thus, β cannot be orthonormal either.

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3 (h) Since $[\vec{v}]_\beta = \langle -3, 20 \rangle$ we know

that $\vec{v} = -3\vec{a} + 20\vec{b}$.

Thus, $\vec{v} = -3\langle 1, 1 \rangle + 20 \langle 1, 0 \rangle = \langle -3 + 20, -3 \rangle = \langle 17, -3 \rangle$

So, $\boxed{\vec{v} = \langle 17, -3 \rangle}$

4 (a) We want to solve $c_1\vec{i}+c_2\vec{j}+c_3\vec{k}=\vec{0}$.

This gives $c_1\langle 1,0,0\rangle + c_2\langle 0,1,0\rangle + c_3\langle 0,0,1\rangle = \langle 0,0,0\rangle$.

So, $\langle c_1, c_2, c_3\rangle = \langle 0,0,0\rangle$.

Thus,

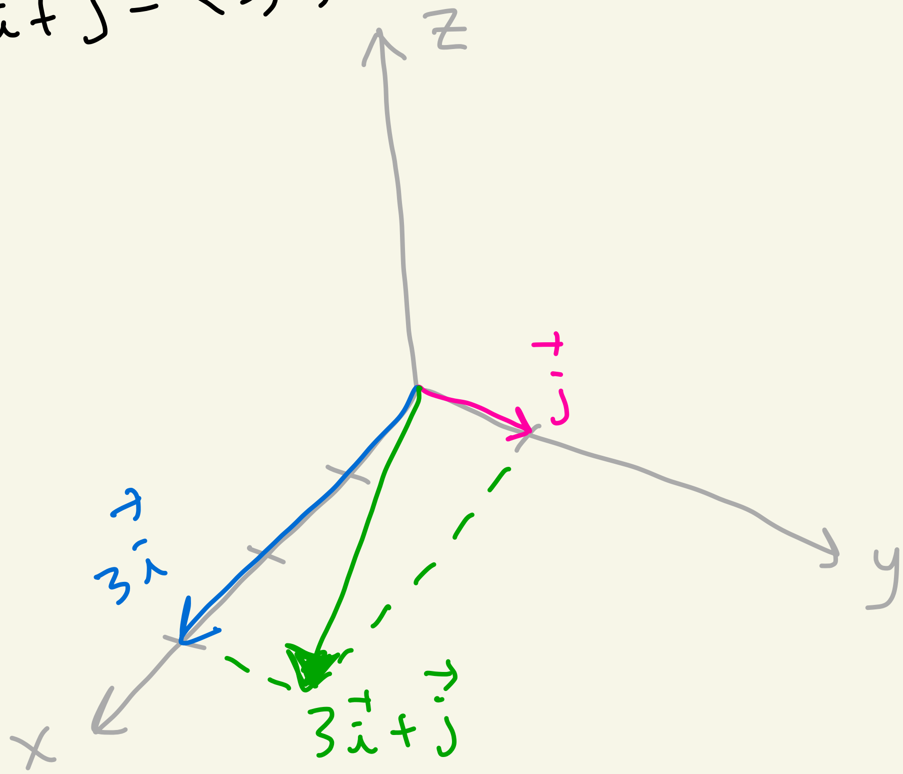
$c_1 = 0, c_2 = 0, c_3 = 0$

is the only solution to $c_1\vec{i}+c_2\vec{j}+c_3\vec{k}=\vec{0}$.

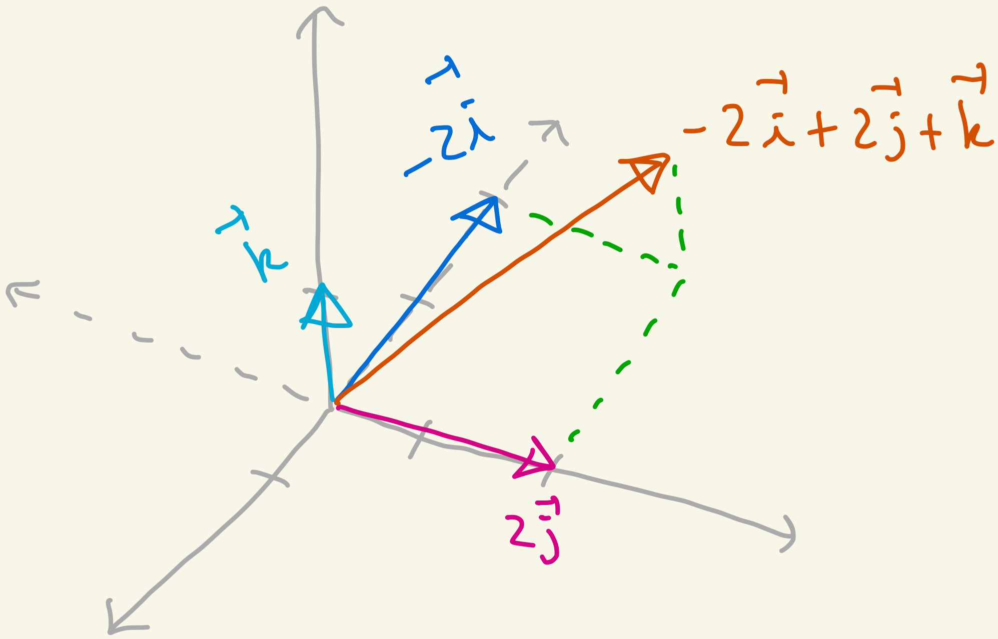
So, $\vec{i},\vec{j},\vec{k}$ are linearly independent.

Since we have 3 lin. ind. vectors in $\mathbb{R}^3$ we know that $\beta=[\vec{i},\vec{j},\vec{k}]$ is a basis for $\mathbb{R}^3$.

4(b) $3\vec{i}+\vec{j}=\langle 3,1,0\rangle$



④(c) $-2\vec{i}+2\vec{j}+\vec{k} = \langle -2, 2, 1\rangle$



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④(d) $\vec{v} = \langle -1, 2, 1\rangle = -\vec{i}+2\vec{j}+\vec{k}$

$[\vec{v}]_\beta = \langle -1, 2, 1\rangle$

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④(e)

$\vec{i}\cdot\vec{j} = \langle 1,0,0\rangle\cdot\langle 0,1,0\rangle = 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0$

$\vec{j}\cdot\vec{k} = \langle 0,1,0\rangle\cdot\langle 0,0,1\rangle = 0\cdot 0 + 1\cdot 0 + 0\cdot 1 = 0$

$\vec{i}\cdot\vec{k} = \langle 1,0,0\rangle\cdot\langle 0,0,1\rangle = 1\cdot 0 + 0\cdot 0 + 0\cdot 1 = 0$

Thus, $\beta = [\vec{i}, \vec{j}, \vec{k}]$ is an orthogonal basis.

Since

$$\|\vec{i}\| = \sqrt{1^2+0^2+0^2} = 1$$

$$\|\vec{j}\| = \sqrt{0^2+1^2+0^2} = 1$$

$$\|\vec{k}\| = \sqrt{0^2+0^2+1^2} = 1$$

we have that $\beta = [\vec{i}, \vec{j}, \vec{k}]$ is an orthonormal basis.

$\boxed{4(f)}$

Since we have an orthonormal basis we have that

$$\vec{v} = \underbrace{(\vec{v}\cdot\vec{i})}_{c_1}\vec{i} + \underbrace{(\vec{v}\cdot\vec{j})}_{c_2}\vec{j} + \underbrace{(\vec{v}\cdot\vec{k})}_{c_3}\vec{k}$$

And

$$\vec{v}\cdot\vec{i} = \langle 6,1,-5\rangle\cdot\langle 1,0,0\rangle = 6+0+0 = 6$$

$$\vec{v}\cdot\vec{j} = \langle 6,1,-5\rangle\cdot\langle 0,1,0\rangle = 0+1+0 = 1$$

$$\vec{v}\cdot\vec{k} = \langle 6,1,-5\rangle\cdot\langle 0,0,1\rangle = 0+0-5 = -5$$

So,

$$\vec{v} = 6\vec{i} + \vec{j} - 5\vec{k}$$

Thus, $\boxed{[\vec{v}]_\beta = \langle 6,1,-5\rangle}$

5 (a)

Need to solve

$$c_1\vec{a} + c_2\vec{b} + c_3\vec{c} = \vec{0}$$

which is

$$c_1\langle 1,1,0\rangle + c_2\langle -1,1,0\rangle + c_3\langle 0,0,1\rangle = \langle 0,0,0\rangle$$

which gives

$$\langle c_1,c_1,0\rangle + \langle -c_2,c_2,0\rangle + \langle 0,0,c_3\rangle = \langle 0,0,0\rangle$$

which gives

$$\langle c_1-c_2, c_1+c_2, c_3\rangle = \langle 0,0,0\rangle$$

This gives

$$\boxed{\begin{aligned} c_1 - c_2 \quad &= 0 \\ c_1 + c_2 \quad &= 0 \\ c_3 &= 0 \end{aligned}}$$

Solving:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{-R_1+R_2\to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\frac{1}{2}R_2\to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

So we get

$$\boxed{\begin{aligned} c_1 - c_2 \quad &= 0 \quad (1) \\ c_2 \quad &= 0 \quad (2) \\ c_3 &= 0 \quad (3) \end{aligned}} \longrightarrow \boxed{\begin{aligned} &(3)\ c_3 = 0 \\ &(2)\ c_2 = 0 \\ &(1)\ c_1 = c_2 = 0 \end{aligned}}$$

no free variables

Since the only solution to

$$c_1\vec{a}+c_2\vec{b}+c_3\vec{c}=\vec{0}$$

is $c_1=0, c_2=0, c_3=0$ we know that $\vec{a},\vec{b},\vec{c}$ are linearly independent.

Since β consists of 3 lin. ind. vectors in $\mathbb{R}^3$ we know that β is a basis for $\mathbb{R}^3$.

$\boxed{5(b)}$ Since $[\vec{v}]_\beta = \langle 3,1,-4\rangle$ we know that

$$\begin{aligned}\vec{v} &= 3\vec{a}+\vec{b}-4\vec{c}\\ &= 3\langle 1,1,0\rangle + \langle -1,1,0\rangle - 4\langle 0,0,1\rangle\\ &= \langle 3,3,0\rangle + \langle -1,1,0\rangle + \langle 0,0,-4\rangle\\ &= \langle 3-1, 3+1, -4\rangle\\ &= \langle 2,4,-4\rangle\end{aligned}$$

$\boxed{5(c)}$ Need to solve $\vec{v} = c_1\vec{a}+c_2\vec{b}+c_3\vec{c}$

This gives $\langle 3,3,2\rangle = c_1\langle 1,1,0\rangle + c_2\langle -1,1,0\rangle + c_3\langle 0,0,1\rangle$

So, $\langle 3,3,2\rangle = \langle c_1-c_2, c_1+c_2, c_3\rangle$

Thus, $\boxed{\begin{aligned} c_1 - c_2 \quad &= 3 \\ c_1 + c_2 \quad &= 3 \\ c_3 &= 2 \end{aligned}}$

Solving:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{-R_1+R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_2 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Thus,

$$\boxed{\begin{aligned} c_1 - c_2 \quad &= 3 \\ c_2 \quad &= 0 \\ c_3 &= 2 \end{aligned}} \quad \begin{aligned} &1 \\ &2 \\ &3 \end{aligned}$$

So, 3 $c_3 = 2$

2 $c_2 = 0$

1 $c_1 = 3 + c_2 = 3 + 0 = 3$.

plug these back into $\vec{v} = c_1 \vec{a} + c_2 \vec{b} + c_3 \vec{c}$ to get

$$\vec{v} = 3\vec{a} + 0\vec{b} + 2\vec{c}$$

Thus, $\boxed{[\vec{v}]_\beta = \langle 3, 0, 2 \rangle}$

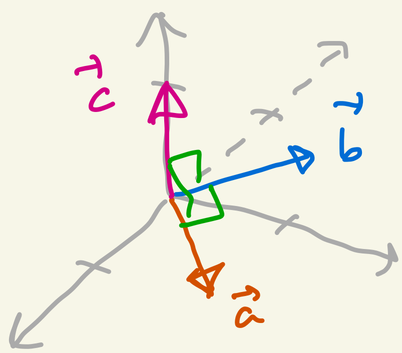
$\boxed{5(d)}$

$\vec{a}\cdot\vec{b} = \langle 1,1,0\rangle \cdot \langle -1,1,0\rangle = -1+1+0 = 0$

$\vec{a}\cdot\vec{c} = \langle 1,1,0\rangle \cdot \langle 0,0,1\rangle = 0+0+0 = 0$

$\vec{b}\cdot\vec{c} = \langle -1,1,0\rangle \cdot \langle 0,0,1\rangle = 0+0+0 = 0$

Thus, $\beta = [\vec{a},\vec{b},\vec{c}]$ is an orthogonal basis.

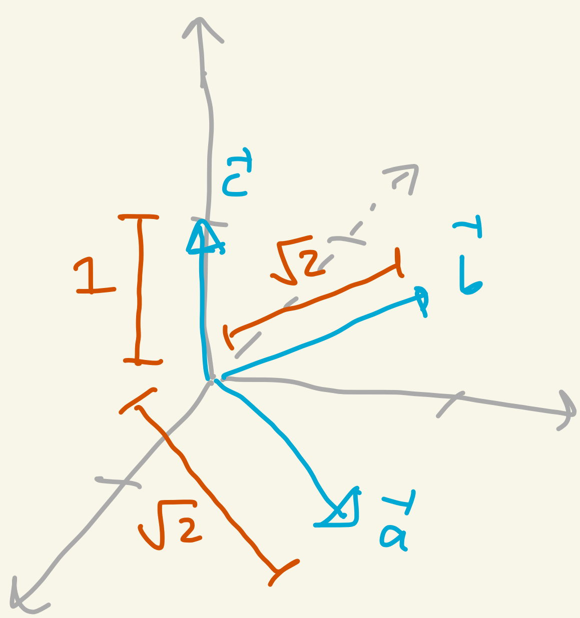


However,

$$\left.\begin{aligned} \|\vec{a}\| &= \sqrt{1^2+1^2+0^2} = \sqrt{2} \\ \|\vec{b}\| &= \sqrt{(-1)^2+1^2+0^2} = \sqrt{2} \end{aligned}\right\} \text{not } 1$$

$$\|\vec{c}\| = \sqrt{0^2+0^2+1^2} = \sqrt{1} = 1$$

So, β is not an orthonormal basis.



$\boxed{⑤(e)}$ Since we have an orthogonal basis we know that

$$\vec{v} = \underbrace{\left(\frac{\vec{v}\cdot\vec{a}}{\|\vec{a}\|^2}\right)}_{c_1}\vec{a} + \underbrace{\left(\frac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|^2}\right)}_{c_2}\vec{b} + \underbrace{\left(\frac{\vec{v}\cdot\vec{c}}{\|\vec{c}\|^2}\right)}_{c_3}\vec{c}$$

We get

$$\frac{\vec{v}\cdot\vec{a}}{\|\vec{a}\|^2} = \frac{\langle 1,2,3\rangle\cdot\langle 1,1,0\rangle}{\left(\sqrt{1^2+1^2+0^2}\right)^2} = \frac{1+2+0}{(\sqrt{2})^2} = \frac{3}{2}$$

$$\frac{\vec{v}\cdot\vec{b}}{\|\vec{b}\|^2} = \frac{\langle 1,2,3\rangle\cdot\langle -1,1,0\rangle}{\left(\sqrt{(-1)^2+1^2+0^2}\right)^2} = \frac{-1+2+0}{(\sqrt{2})^2} = \frac{1}{2}$$

$$\frac{\vec{v}\cdot\vec{c}}{\|\vec{c}\|^2} = \frac{\langle 1,2,3\rangle\cdot\langle 0,0,1\rangle}{\left(\sqrt{0^2+0^2+1^2}\right)^2} = \frac{0+0+3}{(\sqrt{1})^2} = 3$$

Thus,

$$\vec{v} = \frac{3}{2}\vec{a} + \frac{1}{2}\vec{b} + 3\vec{c}$$

So,

$$\boxed{[\vec{v}]_\beta = \left\langle \frac{3}{2}, \frac{1}{2}, 3 \right\rangle}$$

$\boxed{6(a)}$

We need to solve

$$c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 + c_4\vec{e}_4 = \vec{0}$$

This gives

$$c_1\langle 1,0,0,0\rangle + c_2\langle 0,1,0,0\rangle + c_3\langle 0,0,1,0\rangle + c_4\langle 0,0,0,1\rangle = \langle 0,0,0,0\rangle$$

This gives

$$\langle c_1,0,0,0\rangle + \langle 0,c_2,0,0\rangle + \langle 0,0,c_3,0\rangle + \langle 0,0,0,c_4\rangle = \langle 0,0,0,0\rangle$$

Thus,

$$\langle c_1, c_2, c_3, c_4\rangle = \langle 0,0,0,0\rangle.$$

So, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$.

Since the only solution to

$$c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 + c_4\vec{e}_4 = \vec{0}$$

is $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$ we know that $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ are linearly independent.

Since we have 4 lin. ind. vectors in $\mathbb{R}^4$ we know that $\beta = [\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4]$ are a basis for $\mathbb{R}^4$.

$\boxed{6(b)}$ Since $[\vec{v}]_\beta = \langle -3, 1, -4, \pi \rangle$ we know that

$$\vec{v} = -3\vec{e}_1 + 1\cdot\vec{e}_2 - 4\cdot\vec{e}_3 + \pi\cdot\vec{e}_4$$

$$= -3\langle 1,0,0,0\rangle + 1\cdot\langle 0,1,0,0\rangle - 4\cdot\langle 0,0,1,0\rangle + \pi\cdot\langle 0,0,0,1\rangle$$

$$= \langle -3, 1, -4, \pi \rangle$$

$\boxed{6(c)}$

$$\vec{v} = \langle \tfrac{2}{3}, 7, 5, -10 \rangle$$

$$= \langle \tfrac{2}{3}, 0, 0, 0\rangle + \langle 0,7,0,0\rangle + \langle 0,0,5,0\rangle + \langle 0,0,0,-10\rangle$$

$$= \tfrac{2}{3}\langle 1,0,0,0\rangle + 7\langle 0,1,0,0\rangle + 5\langle 0,0,1,0\rangle - 10\langle 0,0,0,1\rangle$$

$$= \tfrac{2}{3}\vec{e}_1 + 7\vec{e}_2 + 5\vec{e}_3 - 10\vec{e}_4$$

Thus,

$$\boxed{[\vec{v}]_\beta = \langle \tfrac{2}{3}, 7, 5, -10 \rangle}$$

$\boxed{6(d)}$

$\vec{e}_1 \cdot \vec{e}_2 = \langle 1,0,0,0\rangle \cdot \langle 0,1,0,0\rangle = 0+0+0+0=0$

$\vec{e}_1 \cdot \vec{e}_3 = \langle 1,0,0,0\rangle \cdot \langle 0,0,1,0\rangle = 0+0+0+0=0$

$\vec{e}_1 \cdot \vec{e}_4 = \langle 1,0,0,0\rangle \cdot \langle 0,0,0,1\rangle = 0+0+0+0=0$

$\vec{e}_2 \cdot \vec{e}_3 = \langle 0,1,0,0\rangle \cdot \langle 0,0,1,0\rangle = 0+0+0+0=0$

$\vec{e}_2 \cdot \vec{e}_4 = \langle 0,1,0,0\rangle \cdot \langle 0,0,0,1\rangle = 0+0+0+0=0$

$\vec{e}_3 \cdot \vec{e}_4 = \langle 0,0,1,0\rangle \cdot \langle 0,0,0,1\rangle = 0+0+0+0=0$

Thus, β is an orthogonal basis.

Also,

$\|\vec{e}_1\| = \sqrt{1^2+0^2+0^2+0^2} = 1$

$\|\vec{e}_2\| = \sqrt{0^2+1^2+0^2+0^2} = 1$

$\|\vec{e}_3\| = \sqrt{0^2+0^2+1^2+0^2} = 1$

$\|\vec{e}_4\| = \sqrt{0^2+0^2+0^2+1^2} = 1$

Thus, β is an orthonormal basis.

$\boxed{7(a)}$

The answer is $\beta = [\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5]$

$\vec{e}_1 = \langle 1, 0, 0, 0, 0 \rangle$

$\vec{e}_2 = \langle 0, 1, 0, 0, 0 \rangle$

$\vec{e}_3 = \langle 0, 0, 1, 0, 0 \rangle$

$\vec{e}_4 = \langle 0, 0, 0, 1, 0 \rangle$

$\vec{e}_5 = \langle 0, 0, 0, 0, 1 \rangle$

$\boxed{7(b)}$

$\beta = [\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n]$

Where $\vec{e}_i$ has a 1 in spot i and 0's everywhere else.
